Bayesian Method for Repeated Threshold Estimation

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Motivation: Perceptual Learning

- Non-stationary thresholds
- Dynamics of learning is important
- Must use naïve observers
- Low motivation → high lapsing rates
- Slow learning → many sessions

- Large volume of low-quality binary data
Objective: Data Reduction
Isn’t This a Solved Problem?

- Up/down (Levitt, 1970)
- PEST (Taylor & Creelman, 1967)
- BEST PEST (Pentland, 1980)
- QUEST (Watson & Pelli, 1979)
- ML-Test (Harvey, 1986)
- Ideal (Pelli, 1987)
- YAAP (Treutwein, 1989)
- and many others…
We Solve a Different Problem

- Standard methods:
  - Adaptive stimulus placement
  - Stopping criterion
  - Threshold estimation

- Our method:
  - Threshold estimation
  - Integrate information across blocks
Weibull Psychometric Function

\[ W(x; \alpha, \beta) = 1 - \exp(-\exp((\log x - \log \alpha) \beta)) \]

\[ P(x; \alpha, \beta, \gamma, \lambda) = \gamma + (1 - \gamma - \lambda)W(x; \alpha, \beta) \]

- Threshold log \( \alpha \)
- Slope \( \beta \)
- Guessing rate \( \gamma \)
- Lapsing rate \( \lambda \)
Two Kinds of Parameters

- Threshold log $\alpha$
- Slope $\beta$
- Guessing rate $\gamma$
- Lapsing rate $\lambda$

Parameters of interest $\theta$

Nuisance parameters $\phi$

The nuisance parameters are harder to estimate but change more slowly than the threshold parameter.
Get the Best of Both Worlds

Use long data sequences to constrain the nuisance parameters; use short sequences to estimate the thresholds.
Joint Posterior of $\theta_k, \phi$

$$p(\theta_k, \phi \mid y_k; y_1 \ldots y_{k-1}, y_{k+1} \ldots y_n) =$$

$$p(y_k \mid \theta_k, \phi)p(\theta_k)p(\phi)\prod_{i \neq k} \int p(\theta_i)p(y_i \mid \theta_i, \phi)d\theta_i$$

- Likelihood of current data
- Priors
- Information about $\phi$ extracted from the other data sets
- Modified prior for the current block
Two-Pass Algorithm

- **Pass 1:** for each block $i$, calculate

  $$p(\phi \mid y_i) = \int p(\theta) p(y_i \mid \theta, \phi) d\theta$$

- **Pass 2:** for each block $k$, calculate

  $$p(\theta_k, \phi \mid y_k) = p(y_k \mid \theta_k, \phi) p(\theta_k) p(\phi) \prod_{i \neq k} \int p(\phi \mid y_i)$$
Posterior Thresholds

\[ p(T_k) = \int P^{-1}(0.75; \theta_k, \phi) p(\theta_k, \phi | y_k) d\theta_k d\phi \]
Some Details

- Vaguely informative priors:
  
  \[
p(\log \alpha) \propto N(\mu_\alpha, \sigma_\alpha)
  
  p(\beta) \propto N(\mu_\beta, \sigma_\beta)
  
  p(\lambda) \propto \text{Beta}(a_\lambda, b_\lambda)
  \]

- Implemented on a grid: \( \log \alpha \times \beta \times \lambda \)

- Assume \( \gamma = .5 \) for 2AFC data

- MATLAB software available at
  
  http://alexpetrov.com/softw/
Simulation 1: Stationary

\[ \log \alpha = -1.204 = \text{const} \]

\( \beta = 1.5 \)

\( \lambda = .10 \)

\[ T_{75} = -1.217 \]
Stimulus Placement

- 2 interleaved staircases
- 100 trials/block
  - 10 catch
  - 40 x 3down/1up
  - 50 x 2down/1up
- 100 runs of 12 blocks each

http://alexpetrov.com/pub/vss06/
# Threshold Estimators

<table>
<thead>
<tr>
<th>Estimator</th>
<th>Mean</th>
<th>Med</th>
<th>Std</th>
</tr>
</thead>
<tbody>
<tr>
<td>ML</td>
<td>-1.24</td>
<td>-1.23</td>
<td>.27</td>
</tr>
<tr>
<td>Median</td>
<td>-1.26</td>
<td>-1.23</td>
<td>.28</td>
</tr>
<tr>
<td>Mean</td>
<td>-1.30</td>
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<td>.31</td>
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<tr>
<td>Std. dev.</td>
<td>0.41</td>
<td>0.36</td>
<td>.15</td>
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</tbody>
</table>

1200 Monte Carlo estimates
True 75% threshold = -1.217
$\beta \times \lambda$ Distribution from Pass 1

- Slope $\beta$
- Lapsing rate $\lambda$

- $P(\text{correct})$

- Log intensity

Graphs showing the distribution of $\beta$ and $\lambda$ and their relationship with $P(\text{correct})$ for both true values and ML estimates.
Catch Trials Are Worthwhile

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<td>-1.24</td>
<td>-1.22</td>
<td>.31</td>
</tr>
<tr>
<td>Median</td>
<td>-1.29</td>
<td>-1.26</td>
<td>.30</td>
</tr>
<tr>
<td>Mean</td>
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<td>-1.33</td>
<td>.34</td>
</tr>
<tr>
<td>Std. dev.</td>
<td>0.58</td>
<td>0.57</td>
<td>.16</td>
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1200 Monte Carlo estimates
No catch trials presented
True 75% threshold = -1.217
Simulation 2: With Learning

\[
\log \alpha = -0.693 \left( e^{-t/800} - 2 \right)
\]

\[
\beta = 1.5
\]

\[
\lambda = .10
\]
Group Learning Curve, N=100
More Realistic Sample, N=10
Individual Runs

http://alexpetrov.com/pub/vss06/
The Method Performs Well

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<tbody>
<tr>
<td>ML</td>
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<td>-0.02</td>
<td>.28</td>
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<tr>
<td>Median</td>
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<td>-0.03</td>
<td>.29</td>
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<tr>
<td>Mean</td>
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<td>.32</td>
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<tr>
<td>Std. dev.</td>
<td>0.42</td>
<td>0.39</td>
<td>.15</td>
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6000 Monte Carlo estimates  
Similar to the stationary case  
No systematic bias over time
Example: Actual Data, N=8

Jeter, Dosher, Petrov, & Lu (2005)
Future Work

- Sensitivity to priors?
- Compare with standard ML methods
- Individual differences
- Estimate slope in addition to threshold
- Non-stationary $\beta$ and $\lambda$?
- Recommended stimulus placement?
- Hierarchical models
The End